*This practice plan was created by **Tyler Erb**, a math teacher and coach at Community House Middle School. Tyler created numerous free resources for MATHCOUNTS coaches in his role as the 2021-2022 DoD STEM Ambassador for MATHCOUNTS. Find more resources and information at **dodstem.us**.*

Mass Points

Warm-Up!

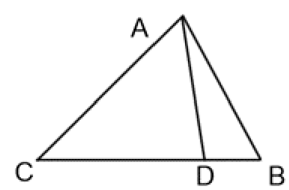
Try these problems before watching the lesson.

Note: *The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. Triangle ABC has ***medians*** AE and BF. The point that they intersect is point G. What is the ratio of the length BG to BF? Express your answer as a ***common fraction***.

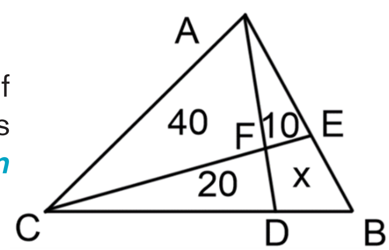
   We know that the point where two medians intersect in a triangle is called a centroid, and it breaks the median into a 2:1 ratio from the vertex. This means BG:GF = 2:1, so BG:BF = $\frac{2}{2+1}$ **or 2/3.**

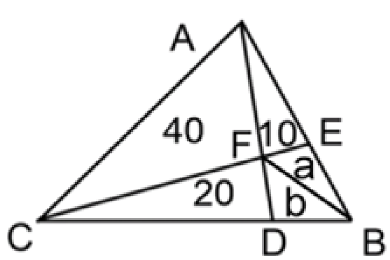
2. CD is 3 times as long as BD. If the area of triangle ABC is 28 m^2, what is the area of triangle ABD?



   Triangle ABC can be split into two triangles, ADC and ABD. Both triangles share the same base, and because they both share A, they have the same height. This means we can split the area of triangle ABC based on the ratio of the bases. CD is 3 times as long as BD, so we know ABD should be $\frac{1}{3+1}=\frac{1}{4}$ **of the whole area. Therefore, the area is 7 m^2.**

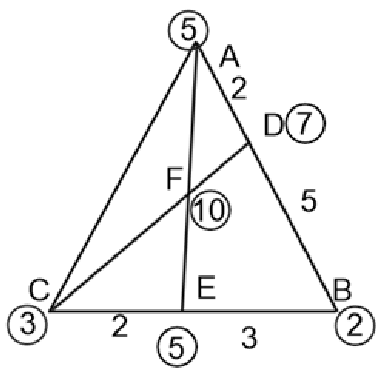
3. CD:DB is a 7 to 3 ratio. The area of triangle AFC is 40 in^2; the area of triangle FEA is 10 in^2; and the area of triangle DFC is 20 in^2. What is the area of quadrilateral FEBD? Express your answer as a ***common fraction.***





   Students may try this problem again using Mass Points. We can split the area of FEBD into two separate triangles and use ratios of bases to create a system of equations. Use [AFB] to represent the area of triangle AFB. We know that $\frac{[AFB]}{[AFC]}=\frac{[DFB]}{[DFC]}=\frac{DB}{CD}=\frac{a+10}{40}=\frac{b}{20}$**.**
   We can use the same process with triangles FCA and FCB to get $\frac{[FCA]}{[FCB]}=\frac{[FEA]}{[FEB]}=\frac{BE}{EA}=\frac{b+20}{40}=\frac{a}{10}$ **. Using both of these ratios, we get the following two equations: 20*a* + 200 = 40*b* and 10*b* + 200 = 40*a*. Solving using elimination or substitution, we get *a* = 50/7 and *b* = 60/7. Since *X* = *a* + *b*, our total area is 110/7 in^2.**

4. In triangle ABC, there is a point D on segment AB so that AD:DB = 2:5. E is on segment CB such that CE:EB = 2:3. The point where AE and CD intersect is point F. What is the ratio of AF to FE?



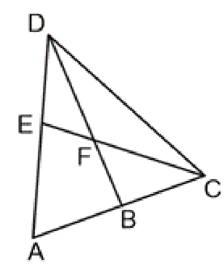
We can use similar triangles to solve this, but Mass Points can be used too. If we know the ratios of lengths of CE:EB is 2:3 and AD:DB = 2:5, we can assign a mass of 2 at B to balance out both CE and AD. This allows us to use a mass of 3 at C, which we get from EB, and 5 at A, which we get from DB. We know that E must have a mass of 5 as it is the sum of masses C and B. The same can be said for D and the mass of 7 (although this isn't needed to solve the problem, it helps us make sure we did our calculations correctly and that our triangle is balanced). We need the ratio AF to FE, but they are equal! So, the ratio is 1.

***Coach instructions:** After students try the warm-up problems, play the video and have them follow along with the solutions. After watching the video, they may want to go back and try to come up with faster ways to solve the warm-up problems before moving on to the final problem set.*

The Problems

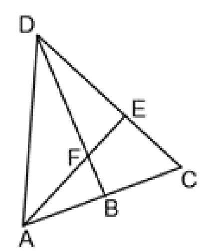
Take a look at the following problems and follow along as they are explained in the video.

5. Point E splits segment DA into a 2 to 3 ratio between DE and EA. AB:BC = 3:5. What is the ratio of DF to FB? Express your answer as a ***common fraction***.



Solution in video. Answer: 16/15.

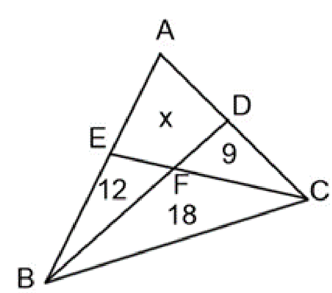
6. Find $\frac{CE}{DE}$ if $\frac{DF}{BF} = 5$ and $\frac{AF}{EF} = 6$. Express your answer as a ***common fraction***.



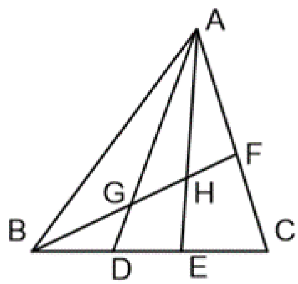
Solution in video. Answer: 7/29.

7. The area of triangle BEF is 12; the area of triangle FBC is 18; and the area of triangle FDC is 9, as shown in the figure. What is the area of quadrilateral EADF? Express your answer as a decimal to the nearest tenth.

Solution in video. Answer: 28.5 units^2.

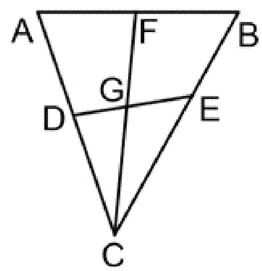


8. As shown in the figure, D and E are points on BC of triangle ABC such that BD:DE:EC = 1:2:3. The ***median*** BF meets AD and AE at G and H, respectively, and is divided into lengths x, y and z. Assuming x, y and z are the smallest possible integers, find $x + y + z$.



Solution in video. Answer: 21.

9. Point D splits segment AC such that $\frac{AD}{DC} = \frac{3}{2}$. F is the ***midpoint*** of segment AB. $\frac{BE}{EC} = \frac{3}{5}$. What is $\frac{DG}{GE}$? Express your answer as a ***common fraction***.



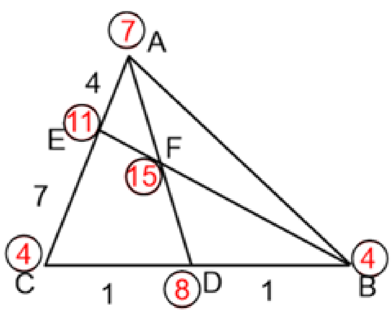
Solution in video. Answer: 16/25.

Piece It Together

Coach instructions: *After watching the video, give students 15-20 minutes to try the next five problems.*

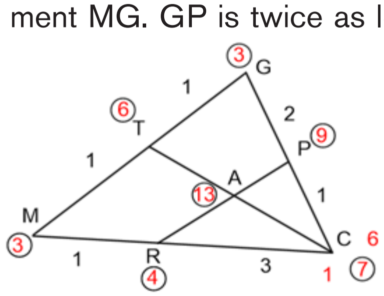
Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

10. In triangle ABC, D is the ***midpoint*** of BC. E splits segment AC such that AE:EC = 4:7. The point where BE and AD intersect is point F. What is the ratio of AF to AD? Express your answer as a ***common fraction***.



First, we draw a diagram of the triangle. With D as a midpoint, we know that CD:DB = 1:1. With AE being a ratio of 4, and DB 1, we assign C a mass of 4 as 4 is the least common multiple (LCM) of 1 and 4. This means we also must have a mass of 4 at B to balance BC, and D has a mass of 8. We assign a mass of 7 at A and 11 at E to balance AC. The mass of F is 15, and that is true whether we look at AD or BE, so we know we did the problem correctly! We are asked for the ratio of AF to AD. Since AF is 8 and AD is 15, the ratio is 8/15.

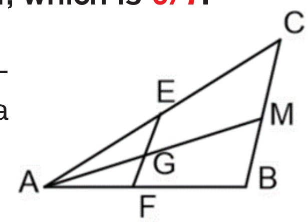
11. In triangle GCM, point P lies on segment GC; point R lies on segment CM; and point T lies on segment MG. GP is twice as long as PC, and MR is 1/3 the length of RC. The point where ***median*** TC and segment PR intersect is point A. What is CA:AT? Express your answer as a ***common fraction***.

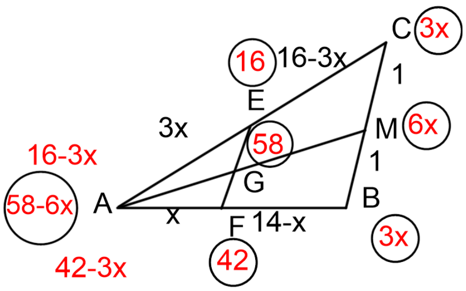


First, we draw a diagram. We can tell we must split the mass here because PR isn't a cevian. TC is a median, which means T is a midpoint of MG, so MT:TG is 1:1. We must balance M and G using the ratios from PC and RC and all of MG. With RC's ratio

being 3, PC's being 1, and MT and TG both being 1, we set both M and G with a mass of 3 as that's the LCM of all. This means C from GC must have a mass of 6, or double the ratio of GP. MR gives C a mass of 1, and therefore, C must have a total mass of 7, or the sum of C from both sides. P has a mass of 9; R has a mass of 4; and T has a mass of 6 from the sum of the masses on either of their respective side lengths. This leaves us with A having a mass of 13, which is balanced on TC and PR. The problem wants CA:AT, which is **6/7**.

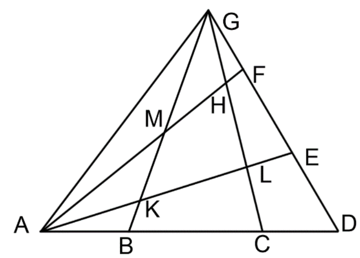
12. M is the ***midpoint*** of BC, AB = 14, and AC = 16. Segment AM and EF intersect at G. If AE = 3AF, then what is EG/GF? Express your answer as a ***common fraction***.





We don't know any ratios other than CM:MB = 1:1. We also don't know the ratio of AF:FB or AE:EC. Instead, we set AF as *x* and AE as 3*x* because AE = 3AF. We can then set FB as 14 − *x*, since AB = 14, and EC = 16 − 3*x*, since AC = 16. To balance BC, we take the LCM from AF, AE, CM and MB, which is 3*x*. This means A from AB is 42 − 3*x*, and A from AC = 16 − 3*x*. Due to the masses at C and B, M = 6*x*. E has a mass of 16 from A and C, and F has a mass of 42 from A and B. We also know that A must be 58 − 6*x*, as it's the sum of the two masses. This gives us a mass of 58 at G, which is balanced from EF and AM. It asks for EG/GF, which is 42/16 and can be simplified to 21/8.

13. In the diagram, AM:MH:HF = 5:3:1. We also know GM:MK:KB = 6:2:1. What is FE:ED? Express your answer as a ***common fraction***.



This is a classic case of too much information. We will never use GC as it doesn't have any given ratios, and it won't help us find any ratios in GD. We have too many cevians to use mass points at first. If we break the triangle up as we do in Figure 1, we can create a ratio for FG:FD to help us solve the problem. This will then give us the ability to use either FE or ED as *x*, and solve for its respective ratio length. In Figure 2, to balance AF and GB, we need to have a central mass of 9 at M. We assign the rest of the masses as shown in the figure and find them balanced. The ratio FG:FD = 2:3 (our diagram is definitely not to scale). Now, we will use AE to help break FD into FE and ED. FE:ED = *x*:(3 − *x*). We will also keep the ratio of AB:BD = 1:2 to help us solve for *x*. Without that ratio as well, we have a balanced triangle, but can't solve for *x*. Keeping segment GB for another ratio, we create Figure 3. In Figure 4, we assign the masses. B has to have a mass of 24 to balance out the 8 from GK and the 3 coming from segment AD. Multiplying all the ratios on AD by 8, we get a mass of 8 at D and 16 at A. We multiply KB by 3 to put a mass of 3 at G. We get a summed mass of 11 at E and 27 at K, which is balanced on both GB and AE. Next, we solve for *x*. GE:ED = (2 + *x*):(3 − *x*) = 8:3. We can set up 8/3 = (2 + *x*)/(3 − *x*), and solve for *x* such that *x* = 18/11. However, we aren't done yet! The problem asked for FE:ED, which equals *x*:(3 − *x*). Substituting *x* back in, we simplify and find that FE:ED = 6/5.

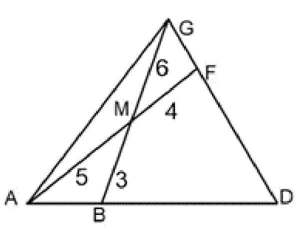


Figure 1

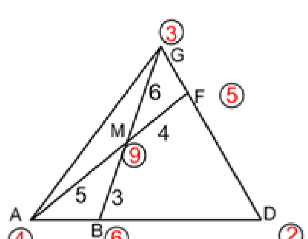


Figure 2

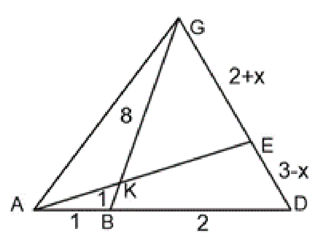


Figure 3

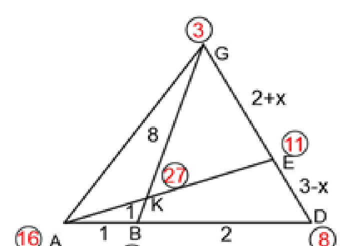
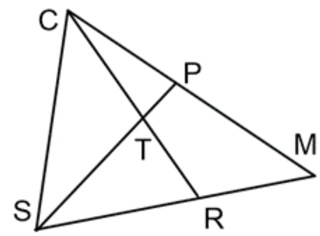


Figure 4

14. Using the diagram, the area of triangle CTS is 21 in^2, while the area of triangle STR is 14 in^2. The area of triangle CPT is 12 in^2. What is the area of quadrilateral PMRT? Express your answer as a ***common fraction***.



In Figure 1, we fill in the areas and create ratios for the bases. Triangle CTS and STR have a shared base length of CR, so we know CT:TR = 21:14 or, simplified, 3:2. The same can be said for triangle CPT and CTS both sharing PS. ST:TP = 21:12 or, simplified, 7:4. In Figure 2, we start assigning masses. As T is the balance point between CR and PS, it must be the LCM of the sum of the ratio lengths of CR and PS, which are 5 and 11, respectively. Once we know T is 55, we then multiply the ratio of the masses on CR by 11 and PS by 5. The mass of C is 22, and the mass of P is 35. We can subtract them to get the mass of M is 13. This also works out with the masses on SM, so we know our triangle is balanced. We can then look at the bigger triangles, CPS and PSM, and create a ratio based on their base lengths, CP and PM. $\frac{[SPC]}{[SPM]} = \frac{21+12}{14+x} = \frac{13}{22}$ **where *x* is the area of quadrilateral PMRT. We solve for *x* and get the area of quadrilateral PMRT = 544/13.**

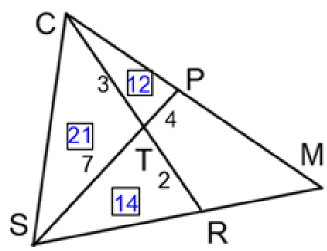


Figure 1

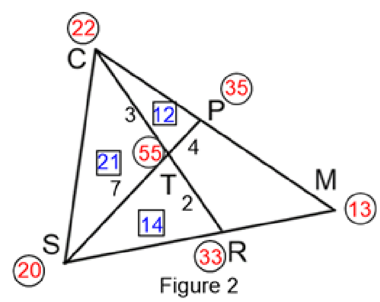


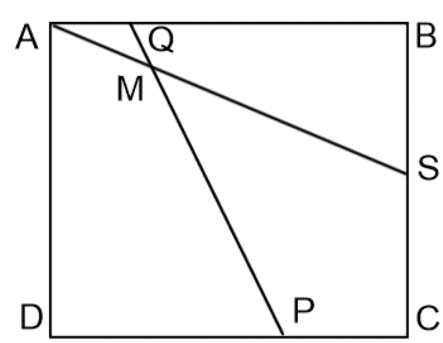
Figure 2

Optional Extension

To extend your understanding and have a little fun with math, try the following activity.

You have applied mass points to triangles, but what would it look like in a quadrilateral? In many geometry problems, we may need to add segments to make our drawing easier to interpret or to use properties of figures that we know. As always, we may use mass points here because the problem gives us ratios of side lengths. Can you figure out what to add to the figure to make it solvable?

ABCD is a rectangle with an area of 96 in^2. Q lies on AB such that AQ:QB = 1:5, and P lies on CD such that CP:PD = 1:3. S is the midpoint of BC. M is the intersection of AS and QP. AB is 12 inches, and AD is 8 inches. Find the area of quadrilateral MSCP. Express your answer as a common fraction.



We have ratios, so we can still use mass points! We will extend out AB to point N (as shown in Figure 1), such that triangle BNS is congruent to triangle CPS by Angle-Side-Angle. We also create segment PN and AP, so that we have triangle APN with two cevians. All numbers in Figure 1 are both lengths and ratios of their respective sides. With triangle APN, we can split it into smaller triangles and compare the ratios of their bases to the area. In Figure 2, we assign our masses to the vertices and balance the triangle. The area of triangle APN is 60 in^2 due to a base length of 15 on AN and a height of 8. Next, we look at triangle APN with PN as the base. With S as a midpoint, that means the area of ASP is equal to the area of ASN, or each triangle has an area of 30 in^2. We can use the ratios of AM:MS to find the areas of triangles AMP and SMP. AM:MS = 20:65 = 4:13, or SMP is 13/17 of ASP. The area of triangle SMP is 390/17, and the area of triangle CPS is 6, so the area of quadrilateral MSCP is **492/17**.

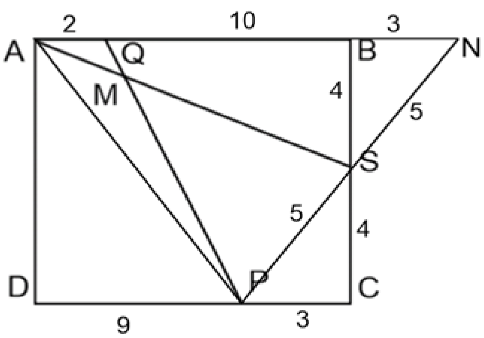


Figure 1

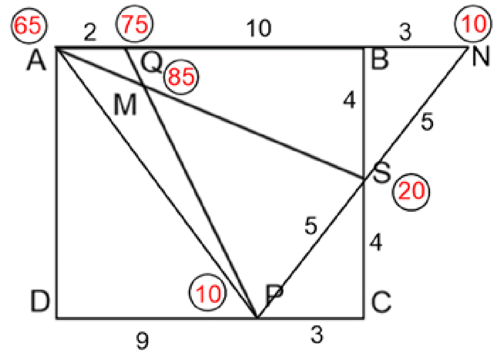


Figure 2