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This practice plan was created by ***Tyler Erb****, a math teacher and coach at Community House Middle School. Tyler created numerous free resources for MATHCOUNTS coaches in his role as the 2021-2022 DoD STEM Ambassador for MATHCOUNTS. Find more resources and information at* ***dodstem.us****.*

Basics of Bases

Warm-Up!

Try these problems before watching the lesson.

Note: *The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. Rewrite 123 to show it as multiples of powers of 10.

   We can rewrite 123 as $1(10^2) + 2(10^1) + 3(10^0)$.

2. Rewrite the number $12.\overline{34}$ as a ***common fraction***.

   To convert the number to an improper fraction, we must first convert the decimal to a fraction. We set $x = .343434...$, which means $100x = 34.343434...$. Subtracting x from $100x$, we get $99x = 34$, which means $x = 34/99$. To convert the whole number to an improper fraction, we change 12 to 1188/99 and add to 34/99, which has a sum of 1222/99.

3. What is the sum of the solutions to the equation $(2x + 3)^2 = 6x^2 + 7x + 11$? Express your answer as a ***common fraction***.

   Vieta's Formula would be quickest here, but let's factor for the sake of later problems. First, we can expand $(2x + 3)^2$ to get $4x^2 + 12x + 9$. We can subtract this from both sides of the equation given in the problem to get $0 = 2x^2 - 5x + 2$. We can factor this to $(2x - 1)(x - 2) = 0$. Therefore, $2x - 1 = 0$ or $x - 2 = 0$, so $x = 1/2$ or $x = 2$. The sum of the solutions is 5/2.

4. How many 3-digit numbers are there?

   The largest 3-digit number is 999 and the smallest is 100. We subtract and then add 1 to find that there are 900 3-digit numbers.

The Problems

Take a look at the following problems and follow along as they are explained in the video.

Coach instructions: *After students try the warm-up problems, play the video and have them follow along with the solutions. After watching the video, they may want to go back and try to come up with faster ways to solve the warm-up problems before moving on to the final problem set.*

5. a) Convert 233_5 to base 10. b) Convert $233_{5/2}$ to base 10.

Solution in video. Answer: a) 68 b) 23.

6. Convert 254_6 to base 4.

Solution in video. Answer: $1{,}222_4$.

7. What is the ***product*** of 52_6 and 43_6? Write your answer in base 6.

Solution in video. Answer: $4{,}000_6$.

8. If $a = .\overline{4}_6$ and $b = 12.\overline{3}_5$, what is their ***product*** in base 4? Express your answer as a ***common fraction***.

Solution in video. Answer: $\left(\frac{133}{11}\right)_4$.

9. Assuming b is positive, what is b if $(13_b)^2 = 202_b$?

Solution in video. Answer: 7.

Piece It Together

Coach instructions: *After watching the video, give students 15-20 minutes to try the next seven problems.*

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

10. What is the sum of…

a) 35_6 and 42_6? b) 723_8 and 462_8?

a) We will first add up the units digit which gives us 7. However, because we are in base 6, 7 = 11_6 or one group of 6 with a remainder of 1. We carry the 1 and 3 + 4 + 1 = 8, which is 12_6. Therefore, our final answer is 121_6.

b) We add the 3 and 2 to get 5, which is not bigger than 8, so we are fine. Then 6 + 2 = 8, which is 10_8, as there is one group of 8 with no remainder. We carry the 1 and now do 1 + 7 + 4, which equals 12_{10} or 14_8. Therefore, our final answer is $1{,}405_8$.

11. What is the ***product*** of 72_8 and 63_8? Write the number in base 6.

First, we need to find the product of the two numbers. Remember, anytime we get a group of 8, we can carry a 1. To the right is the multiplication of the 2 numbers in base 8. Notice when we multiply 3 and 7, we get 21 in base 10. However, this would be 2 groups of 8 with a remainder of 5. The same happens with the product of 6 and 2, which is 12, but is 14 in base 8. After we get the two products, we will add the two numbers. When adding, we can also carry 1 whenever we have a group of 8. For example, 5 and 4 would be 9 in base 10, but is 11 in base 8. Now that we know our product, we must convert it to base 6. It's easier to first convert to base 10 and then do repeated division for base 6: $5(8)^3 + 6(8)^2 + 1(8) + 6 = 2{,}958$ in base 10. Finally, we convert to base 6 by repeatedly dividing by 6 and writing down the remainder as the digit, which gives us $21{,}410_6$.

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   72
 x 63
  256
+ 5340
  5616
      8
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12. How many 3-digit numbers in base 6 are also 3-digit numbers in base 8?

We first must realize that being the smallest 3-digit base 8 number will be the lower end of our bound, and our upper bound is the largest 3-digit base 6 number. The smallest base 8 3-digit number is 100_8 or $1(8)^2$, which is 64. The largest base 6 3-digit number is 555_6, which is $5(6^2) + 5(6^1) + 5(6^0) = 215$. An easier way to find this would be one less than the smallest 4-digit number, which would be 1000_6 or $1(6^3)$ and take 1 away. We want to know how many 3-digit numbers there are, so we take 215 – 64 and then add 1 (because we want to know how many total numbers, not just the difference). Our final answer is 152.

13. What is b such that the ***product*** of 34_b and 51_b is $2{,}424_b$? Base b is a positive ***integer***.

By transforming these into polynomials, the problem can be solved without using much guess and check. We know the number must also be bigger than 5 as that is the largest digit, but if it isn't close to 5, this will take too much guess and check work. 34_b becomes $3b + 4$, 51_b is $5b + 1$, and $2{,}424_b$ becomes $2b^3 + 4b^2 + 2b + 4$. We can use these expressions to solve for b: $(3b + 4)(5b + 1) = 2b^3 + 4b^2 + 2b + 4$ or $15b^2 + 23b + 4 = 2b^3 + 4b^2 + 2b + 4$. Manipulate the equation to get $2b^3 - 11b^2 - 21b = 0$. Factoring this, we get

$b(2b + 3)(b - 7) = 0$. We know that b must be a positive integer, so therefore b = 7.

14. b is the smallest possible ***integer*** base for which 232_b is a square number in base 10. What is the ***square root*** of 232_b expressed in base b?

With 3 being the largest digit in the number, we know $b \geq 4$. Doing some minor casework, we find if $b = 4$, $232_b = 46_{10}$; if $b = 5$, $232_b = 67_{10}$; if $b = 6$, $232_b = 92_{10}$; and finally, if $b = 7$, $232_b = 121_{10}$. This means that $b = 7$. The square root of 121 is 11 and expressed in base 7, we would get 14.

15. What is the smallest ***integer*** in base 10 that would have 5 digits in base 3/2?

Your first assumption may be to write $10{,}000_{3/2}$. However, this is not an integer! We instead do the casework on the right. Every time we get a group of 3 in a lower digit place, we carry over 2, which is why $3 = 20_{3/2}$. The same happens from 5 to 6. We add 1 to the units digit, which would give us $40_{3/2}$, but that can't exist because the digit is higher than 3. Instead, we change the 4, as shown in the chart to the right, so that it is $210_{3/2}$. Also, we found that 9 is the smallest 4-digit number. If we combine the 9 and 6, we will get a group of 3 in the 3rd digit place, which will then have us carry 2 into the 4th digit place, making it 4010, which again can't exist. Instead, it turns into $21{,}010_{3/2}$, which is 15.

Base 10	Base 3/2
1	1
2	2
3	20
4	21
5	22
6	210
7	211
8	212
9	2100

Optional Extension

To extend your understanding and have a little fun with math, try the following activity.

So far, we have talked about fractional bases and positive integer bases, but there are many different types. Let's spend some time looking at negative bases. Keep in mind that each number must be represented in a unique way. This means that you can't use a negative sign in a negative base number.

- Create an addition and multiplication table for base −6. What patterns do you see or what do you notice?
- Simplify the following expressions, with all numbers in base −6:
    1. 234 + 152
    2. 342 + 425
    3. 42 × 53
    4. 123 × 45

a) When dealing with base −6, we know our largest digit possible is 5. Shown are the addition and multiplication tables for base −6. We first notice that as soon as we try to write 6, we end up having to write a three-digit number. This is because the third digit in base −6 is $(-6)^2$ or multiples of 36. We must take $36 + 5(-6)$ to even give us 6. The second piece of information we may notice is that as the number decreases in the second digit spot, it gets larger. The second digit is

multiples of $(-6)^1$, which means we want less negatives for a larger number. We can see that 25 is 121_{-6}, which is larger than 20, which is 132_{-6}. Yes, 121 > 132! To generalize, the odd digit spots will increase the number and the even digit spots will decrease the number. If a number has an odd number of digits, it is positive, and if it has an even number of digits, it is negative.

+	1	2	3	4	5
1	2	3	4	5	150
2	3	4	5	150	151
3	4	5	150	151	152
4	5	150	151	152	153
5	150	151	152	153	154

x	1	2	3	4	5
1	1	2	3	4	5
2	2	4	150	152	154
3	3	150	153	140	143
4	4	152	140	144	132
5	5	154	143	132	121

b) Whenever we get a sum of 6, we can either carry over a 15_{-6}, which means 5 to that digit place and 1 to the left as well, or we can carry over a −1, as each digit place alternates between positive and negative.

1. 234 + 152: Adding 4 and 2 gives us 6, so we will carry −1 to the next digit place. Then, 3 + 5 + (−1) = 7. We will keep a 1 and then carry a −1, because we made 6 groups of $(-6)^1$. Finally, 2 + 1 − 1 = 2. Our final answer is 210_{-6}.
2. 342 + 425: First, we add the units digit, which is 2 + 5 = 7. However, we can only keep the 1 and carry a −1. Next, 4 + 2 + (−1) makes 5, so there is no carry. Finally, 3 + 4 = 7. However, we can't carry a −1 because we are at the end of our addition. Instead, looking back at our multiplication table, we see 7 = 151_{-6}. Our final answer is $15{,}151_{-6}$.
3. 42 × 53: We first multiply 3 and 2 to get 6, which means we will instead have a 0 for the first digit spot and carry a −1. Next, we multiply 3 by the second digit spot, which is 4(3), and subtract 1 for the carry. This gives us 11, which is 155_{-6}. Therefore, 42 multiplied by 3 is $1{,}550_{-6}$. Then, we multiply 42 by 50, which first has us multiply 5 × 2 = 10. We will leave 4 and carry a −1. Next, 5 multiplied by 4 is 20. Subtract the 1 to get 19, and 19 is 131_{-6}. Therefore, 50 multiplied by 43 is $13{,}140_{-6}$. Finally, we add these two numbers up: 13,140 + 1,550 = $14{,}530_{-6}$.
4. 123 × 45: Multiply 5 by 3 to get 15, which means leave 3 in the first digit spot and carry a −2. Then, 5(2) − 2 = 8, which means leave 2 and carry a −1. Finally, 5(1) − 1 = 4. Therefore, 123 multiplied by 5 is 423. Next, we multiply 40 by 123. First, 4 multiplied by 3 is 12, or leave 0 and carry −2. Then, 4(2) − 2 = 6, or leave 0 and carry −1. Finally, 1(4) − 1 = 3. Therefore, the product is 3,000. Finally, we add the two numbers: 3,000 + 423 = $3{,}423_{-6}$.